Wave Energy and Momentum / Wave Kinetics

- have discussed basic waves in uniform plasma, unmagnetized:

EM: $\omega^2 = \omega_{pe}^2 + c^2k^2 = \omega_{pe}^2(1+d_e^2k^2)$

Warm Plasma: $\omega^2 = \omega_{pe}^2(1+\alpha k^2\lambda_{De}^2)$
(Langmuir)

Ion Acoustic: $\omega^2 = k^2c_s^2/1+k^2\lambda_{De}^2$

$c_s^2 = T_e/m_i$

- Now seek general 'Poynting' theorem for plasma waves, especially electrostatic, i.e. a relation of form:

$$\partial_t W + \underline{\nabla}\cdot\underline{S} + Q = 0$$

$W \rightarrow$ wave energy density
$\underline{S} \rightarrow$ Wave energy density flux / Momentum
$Q \rightarrow$ Dissipation

- issue: $\rightarrow$ second order in wave amplitude (i.e. quasilinear)
$\rightarrow$ need include medium energy as well as wave EM fields

In pure EM:

$$\partial_t \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[\frac{c}{4\pi} \underline{E} \times \underline{H} \right]$$

$$+ \underline{E} \cdot \underline{J} = 0$$

– How construct:

ⓐ → can derive via Principle of Least Action, wave Lagrangian Density, leading to Action density equation

ⓑ → can derive by considering build-up of energy content in time, allowing for fast (carrier) and slow space-time dependence.

For ⓑ:

$$\frac{dW}{dt} = \frac{1}{8\pi} \operatorname{re} \left(\underline{E}^* \cdot \frac{d\underline{D}}{dt} \right)$$

Consider:

$$\underline{E} = \underline{E}_0(t, \underline{x})\, e^{i(\underline{k}_0 \cdot \underline{x} - \omega t)}$$

carrier

slow space-time variation

$t \leftrightarrow$ build-up of energy

$\underline{x} \leftrightarrow$ spread of initially local perturbation

envelope

slow $t \rightarrow$ frequency α

slow $\underline{x} \rightarrow$ wave vector $\underline{q}$

so

$$\underline{E} = \sum_{\alpha, \underline{q}} \underline{E}_{0\alpha, \underline{q}} \exp\left[i(\underline{k}_0 + \underline{q}) \cdot \underline{x} - i(\omega_0 + \alpha) t \right]$$

[redacted]

and: $D = \epsilon E$, but ϵ non-local in space-time

$$D(\underline{k}, \omega) = \epsilon(\underline{k}, \omega)\, E(\underline{k}, \omega)$$

so

if $\quad f(\underline{k}, \omega) \equiv -i\omega\, \sigma(\underline{k}, \omega)$

$$\frac{d\underline{P}}{dt} = \sum_{\alpha, \underline{q}} f(\omega_0 + \alpha, \underline{k}_0 + \underline{q})\, e^{i(\underline{q}\cdot\underline{x} - \alpha t)} \underset{\alpha, \underline{q}}{\underline{E}_0} \left[e^{i(\underline{k}_0\cdot\underline{x})} e^{-i(\alpha + \omega_0)t} \right] *$$

then expand: $\quad \alpha \ll \omega_0$, $\quad |\underline{q}| \ll |\underline{k}|$

$$\frac{d\underline{P}}{dt} = \sum_{\alpha, \underline{q}} \left[-i\omega\, \sigma(\underline{k}, \omega) + \alpha \frac{\partial}{\partial\omega}(-i\omega\sigma)\Big|_{\underline{k}_0, \omega_0} + \underline{q}\cdot\frac{\partial}{\partial\underline{k}}(-i\omega\sigma)\Big|_{\underline{k}_0, \omega_0} \right] e^{i(\underline{q}\cdot\underline{x} - \alpha t)} *$$

$$\underline{E}_0 \Big| e^{i(\underline{k}_0\cdot\underline{x} - \omega_0 t)} \Big._{\omega_0, \underline{k}_0}$$

operators act on all to right, so

re-summing series:

$$\frac{d\underline{D}}{dt} = \left[-i\omega\epsilon\,\underline{E}_0(t,\underline{x}) + \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\frac{\partial}{\partial t}\underline{E}_0(t,\underline{x}) - \frac{\partial}{\partial\underline{k}}(\omega\epsilon)\cdot\underline{\nabla}\,\underline{E}_0(t,\underline{x})\right]\exp\left[i\underline{k}_0\cdot\underline{x} - i\omega_0 t\right]$$

so

$$\frac{dW}{dt} = \frac{1}{8\pi}\,\mathrm{re}\left(\underline{E}^*\cdot\frac{d\underline{D}}{dt}\right)$$

and thus:

$$\frac{dW}{dt} = \omega\,\epsilon_{IM}(\underline{k},\omega)\frac{|\underline{E}_0|^2}{8\pi}\Big|_{\underline{k}_0,\omega_0} + \frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\frac{|\underline{E}_0|^2}{8\pi}\right] - \underline{\nabla}\cdot\left[\frac{\partial}{\partial\underline{k}}(\omega\epsilon)\frac{|\underline{E}_0|^2}{8\pi}\right]_{\underline{k}_0,\omega_0}$$

thus have:

$$W = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\underline{k}_0, \omega_0} \left(|E_0|^2/8\pi\right)$$ → total wave energy density

$$\underline{S} = -\frac{\partial}{\partial \underline{k}}(\omega \epsilon)\Big|_{\underline{k}_0, \omega_0} \left(|E_0|^2/8\pi\right)$$ → total wave energy density flux

$$Q = \omega \epsilon_{IM} \left(|E_0|^2/8\pi\right)$$ → energy dissipation rate.

Note: For EM wave:

$$\to W \to \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\underline{k}_0, \omega_0} \left(|E_0|^2/8\pi\right) + \frac{\partial}{\partial \omega}(\omega \mu)\Big|_{\underline{k}_0, \omega_0} \left(|H_0|^2/8\pi\right)$$

$$\cdot\ S \to S + \frac{c}{4\pi}(\underline{E} \times \underline{H})$$

S → momentum on/of medium

$\frac{c}{4\pi}(\underline{E} \times \underline{H})$ → em momentum

Note:

i.) At wave resonance, $\epsilon(\underline{k}_0, \omega_0) = 0$

$$W = \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\underline{k}_0, \omega_k} \left(|E_0|^2/8\pi\right)$$

$$\underline{S} = -\omega_k \frac{\partial \epsilon}{\partial \underline{k}}\Big|_{k\,\omega_k} \left(|E_0|^2/8\pi\right) = -\frac{\partial \epsilon/\partial \underline{k}}{\partial \epsilon/\partial \omega}\Big|_{\omega_k} \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_0|^2}{8\pi}$$

$$Q = \omega_k \epsilon_{I\,k}\Big|_{k\,\omega_k} \left(|E_0|^2/8\pi\right) = +v_{gr} W$$

ii.) $\underline{v}_{gr_k} = \underline{S}/W$

$$= -\left(\partial \epsilon/\partial \underline{k}\right)_{\omega_k} / \partial \epsilon/\partial \omega_{\omega_k}$$

Alternatively, along wave path:

$$d\epsilon = \frac{\partial \epsilon}{\partial \omega} d\omega + \frac{d\epsilon}{d\underline{k}} \cdot d\underline{k} = 0$$

$$\underline{v}_{gr} = -\partial \epsilon/\partial \underline{k} \,/\, \partial \epsilon/\partial \omega\Big|_{\omega_k}$$

Physics of Wave Energy/Momentum

i.) $W = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\underline{k}, \omega_0} \left(|E_0|^2/8\pi\right)$

$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$, for cold plasma

$W = \left(1 + \frac{\omega_p^2}{\omega^2}\right)\Big|_{\omega=\omega_{p0}} \frac{|E_0|^2}{8\pi} = 2 \times \frac{|E_0|^2}{8\pi}$

$= W_{\text{Field}} + W_{\text{sloshing energy}}$ 'Wave' = Field + Particle Motion

Sloshing?

$\frac{1}{2} n_0 M |\tilde{V}|^2 = \frac{n_0}{2} \frac{Z^2}{M^2} \frac{M |E_0|^2}{\omega^2} = \frac{1}{8\pi} \frac{\omega_p^2}{\omega^2} |E_0|^2$

ii.) $S = -\omega (\partial \epsilon / \partial \underline{k}) |E_0|^2 / 8\pi$

$\epsilon = 1 - \omega_{pb}^2/(\omega - k V_b)^2$ (Beam Plasma)

beam speed

$\underline{S} = + \omega \omega_{pb}^2 \, 2\underline{k} V_b / (\omega - k V_b)^3 \Rightarrow \underline{S} \sim \underline{k}$

compressional wave

iii) If cold, collisional plasma

$\to$ collisional drag i.e. neutrals

$$\epsilon = 1 - \omega_p^2/\omega(\omega + i\nu)$$

$$\frac{\partial}{\partial t}\underline{V} + \nu\underline{V} = \frac{\Sigma}{M}\underline{E}$$

etc.

$$\cong 1 - \frac{\omega_p^2(\omega - i\nu)}{\omega(\omega^2 + \nu^2)}$$

$$\epsilon_{IM} = \omega_p^2\nu/\omega(\omega^2 + \nu^2)$$

$$Q = \frac{\omega_p^2\nu}{\omega^2 + \nu^2}\frac{|E|^2}{8\pi}$$

$Q \sim \nu$.

Insert

1) Positive / Negative Energy Waves

$$W = (|E|^2/8\pi)\ \omega_k \partial\epsilon/\partial\omega|_{\omega_k}$$
$$= (|E_k|^2/8\pi)\ \partial(\omega\epsilon)/\partial\omega$$

Contrast

$\to$ cold plasma $\quad \epsilon = 1 - \dfrac{\omega_p^2}{\omega^2}$

$$W_k = (|E_k|^2/8\pi)(1 + \omega_p^2/\omega_k^2)$$
$$= |E_k|^2/4\pi$$

Insert:

- Observe:

$$E_{wave} = \overbrace{\omega_k \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \frac{|E_0|^2}{8\pi}}^{?}$$

Now, semi-classically:

$$E_w = N \omega_k \cancelto{1}{\hbar}$$

$$p_w = N k \cancelto{1}{\hbar}$$

whe $N \equiv$ # waves, # quanta

Dimensionally:

$$\mathcal{E} = N\omega \Rightarrow N \sim \mathcal{E}/\omega$$

$\Rightarrow$ Action density

- For Action density, see Posted Notes from Mechanics.

– Action density $N(\underline{x}, \underline{k}, t)$ satisfies wave kinetic equation:

$$\partial_t N + \underline{v}_{gr} \cdot \underline{\nabla} N - \underline{\partial_x} \omega \cdot \underline{\nabla}_k N = C(N)$$

i.e. $\frac{dN}{dt} = C(N)$

along $\frac{d\underline{x}}{dt} = \underline{v}_{gr}$, $\frac{d\underline{k}}{dt} = -\underline{\partial_x} \omega$

If seek $N(\underline{x}, t)$:

$$\partial_t N + \underline{\nabla} \cdot (\underline{v}_{gr} N) = \int d\underline{k} \, \cancelto{0}{C(N)}$$

for # conserving

Understood $N(\underline{x}, t)$ implies packet.

- $W_k > 0$ ⇒ need put energy into oscillator to excite motion

- kinetic energy → $\frac{1}{2} m |V|^2$
  Potential → $|E|^2/8\pi$ (electrostatic)

<u>equal</u> in simple oscillator..

→ Beam-plasma System

$$\begin{cases} \underline{V} = V_0 \hat{z} + \underline{\tilde{V}} \\ 1D \end{cases}$$

$$\frac{\partial \tilde{V}}{\partial t} + V_0 \frac{\partial}{\partial x} \tilde{V} = +\frac{q}{m} E$$

$$\frac{\partial \tilde{n}}{\partial t} + V_0 \frac{\partial \tilde{n}}{\partial x} = -n_0 \underline{\nabla} \cdot \underline{\tilde{V}}$$

$$\epsilon = 1 - \omega_p^2 / (\omega - k V_0)^2$$

$$\omega = k V_0 \pm \omega_p$$

$$W_k = \omega_k \left. \frac{\partial \epsilon}{\partial \omega} \right|_{\omega_k} (|E_k|^2/8\pi)$$

$$= (k V_0 \pm \omega_p) \frac{2 \omega_p^2}{(\omega - k V_0)^3} (|E_k|^2/8\pi)$$

$$= (k V_0 \pm \omega_p) \frac{2 \omega_p^2}{(\pm \omega_p)^3} (|E_k|^2/8\pi)$$

$$W_y = \frac{(kV_0 \pm \omega_p)}{\pm \omega_p} |E_0|^2/4\pi$$

Note:

- $$W_y = \frac{k\,(V_0 \pm \omega_p/k)}{\pm \omega_p} |E_0|^2/4\pi$$

- i.) + root $\rightarrow$ "fast" wave, $\omega = \omega_p + kV_0$

$$W = \left(\frac{kV_0 + \omega_p}{\omega_p}\right)(\quad) > 0$$

~ positive energy wave

ii.) − root $\rightarrow$ "slow" wave, $\omega = -\omega_p + kV_0$

$$W = \frac{(kV_0 - \omega_p)}{-\omega_p}(\quad) = \frac{\omega_p - kV_0}{\omega_p}(\quad)$$

$$= [(\omega_p - kV_0)/\omega_p](\quad)$$

$\Rightarrow$ $W > 0$ for $kV_0 < \omega_p$

$W < 0$ for $\omega_p < kV_0$!

⊕ Negative energy wave !

What is a negative energy wave ?

→ excited by extraction of energy from system

Contrast: Positive energy wave excited by input of energy into system.

→ excitation for beam $\Rightarrow$ bunching.

→ To excite by extraction, negative energy wave occurs in "active" medium

active medium → motion → beam.
↓
free energy.

→ Active medium suggests free energy available for relaxation ⇒ instability!

How tap? → dissipation! (dissp'n → extr. energy → W<0 wave growth) → couple to positive energy wave! (extr energy ⊖, to ⊕, etc.)

N.B. Negative energy wave excited by extraction of energy from active medium

c.) For destabilization by dissipation

$$\partial_t W_k + \nabla \cdot \underline{S}_k + Q_k = 0$$

if $\nabla \cdot \underline{S}_k \approx 0$ (though radiative damping can destabilize negative energy wave)

⇒

$$2\gamma_k = -Q_k / W_k$$

Now if $W_1 < 0 \Rightarrow$ negative energy

$Q_1 > 0 \Rightarrow$ positive dissipation

$\Rightarrow \gamma_1 > 0$.

Ex: Weak collisional dissipn in beam.

ii) For +, − energy wave coupling:
$\Rightarrow$ beam-plasma system.

Idea is to couple positive energy wave in plasma with negative energy wave in beam.

Ex: consider beam-plasma system:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kV_0)^2} \; ; \quad n_b < n_0$$

$n_b = 0 \Rightarrow$ (+) energy plasma oscillations only.

Beam $\Rightarrow$ negative energy waves for $kV_0 > \omega_{pb}$

Active medium $\Rightarrow$ beam kinetic energy.

Now, for modes:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - k V_0)^2} = 0$$

$n_b \ll n_0$, need $\omega \sim k V_0$ for third term to be relevant

$$1 - \frac{\omega_p^2}{(k V_0)^2} - \frac{\omega_{pb}^2}{\delta^2} = 0$$

$$\delta^2 = \omega_{pb}^2 \Big/ 1 - \frac{\omega_p^2}{(k V_0)^2}$$

$$= \omega_{pb}^2 \Big/ \epsilon_{pl}(k, k V_0)$$

Now, $\delta^2 > 0 \Rightarrow$ frequency shift

$\delta^2 < 0 \Rightarrow \omega = k V_0 \pm i|\delta| \Rightarrow$ growth.

$\delta^2 < 0 \Rightarrow (k V_0)^2 < \omega_p^2$

so $\epsilon(k, kV_0) < 0$

$\Rightarrow$ Bunching instability $\rightarrow$ screening acts to enhance charge perturbation.

Need: $\epsilon < 0 \Rightarrow kV_0 < \omega_p$

but $n_b \ll n \Rightarrow$ easy for $\omega_b < kV_0 < \omega_p$.

Can make more explicit connection to $\oplus, \ominus$ energy.